On the Minimal Polynomial of a Resolvent

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ABSTRACT

We derive the connection between the minimal polynomial of an operator and the minimal polynomial of its resolvent. An analogous result is obtained for the minimal polynomials of the iterated resolvents.

1. THE MINIMAL POLYNOMIAL OF A RESOLVENT

Let A be a linear operator on a finite dimensional space. By $x \mapsto \mu_1(x)$ is denoted its minimal polynomial, and by $x_1 \mapsto R(x_1) := (x_1 I - A)^{-1}$ its resolvent. Let $x \mapsto \mu_2(x_1, x)$ be the minimal polynomial of the resolvent R. Our intention is to derive an expression for the polynomial μ_2 in terms of the polynomial μ_1. The result is given in the following theorem:

THEOREM 1. *We have*

$$\mu_2(x_1, x) = \frac{1}{\mu_1(x_1)} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \mu_1^{(k)}(x_1) x^{m-k}, \tag{1}$$

where m is the degree of the polynomial μ_1.

Proof. Let $\mu_1(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_p)^{m_p} = (x-\sigma_1)\cdots(x-\sigma_m)$ be the factorization of the minimal polynomial, where $\sigma_1 = \cdots = \sigma_{m_1} = \lambda_1$, $\sigma_{m_1+1} = \cdots = \sigma_{m_1+m_2} = \lambda_2, \dots$ are the roots of μ_1. We can take a basis (e) of

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the space X such that A in this basis has the Jordan form

$$A(e)=\sum_{j=1}^{p}\dot{+}(\lambda_j I_j+N_j),$$

where N_j is a nilpotent operator of index m_j (see [1, p. 278]). In this basis the following relation holds:

$$R(x_1)=(x_1I-A)^{-1}=\sum_{j=1}^{p}\dot{+}\left[(x_1-\lambda_j)I_j-N_j\right]^{-1}$$

$$=\sum_{j=1}^{p}\dot{+}\left(\frac{1}{x_1-\lambda_j}I_j+M_j\right),$$

and M_j is again a nilpotent operator of index m_j. Since the minimal polynomial does not depend on a particular choice of a basis, we have

$$\mu_2(x_1,x)=\left(x-\frac{1}{x_1-\lambda_1}\right)^{m_1}\cdots\left(x-\frac{1}{x_1-\lambda_p}\right)^{m_p}$$

$$=\prod_{j=1}^{m}\left(x-\frac{1}{x_1-\sigma_j}\right)$$

$$=\sum_{k=0}^{m}x^{m-k}(-1)^k\sum_{1\leqslant i_1<\cdots<i_k\leqslant m}\frac{1}{(x_1-\sigma_{i_1})\cdots(x_1-\sigma_{i_k})}$$

$$=\sum_{k=0}^{m}x^{m-k}\frac{(-1)^k}{k!}\frac{\mu_1^{(k)}(x_1)}{\mu_1(x_1)},$$

and the theorem is proved. ■

COROLLARY 1. *We have*

$$\mu_2(x_1,x_2)=\frac{1}{\mu_1(x_1)}\sum_{k=0}^{m}\frac{(-1)^k}{k!}\mu_1^{(k)}(0)(x_1x_2-1)^k x_2^{m-k}. \qquad (2)$$

Proof. It is sufficient to put into (1) the expression

$$\mu_1(x_1) = \sum_{k=0}^{m} \frac{1}{k!}\mu_1^{(k)}(0)x_1^k.$$

Then we obtain

$$\mu_2(x_1, x_2) = \frac{1}{\mu_1(x_1)} \sum_{j=0}^{m} \frac{(-1)^j}{j!}$$

$$\times \sum_{k=0}^{m} \frac{1}{k!}\mu_1^{(k)}(0)\frac{k!}{(k-j)!}x_1^{k-j}x_2^{m-j}$$

$$= \frac{1}{\mu_1(x_1)} \sum_{k=0}^{m} \frac{1}{k!}\mu_1^{(k)}(0)x_2^{m-k} \sum_{j=0}^{m} (-1)^j \binom{k}{j} x_1^{k-j}x_2^{k-j}$$

$$= \frac{1}{\mu_1(x_1)} \sum_{k=0}^{m} \frac{(-1)^k}{k!}\mu_1^{(k)}(0)(x_1x_2-1)^k x_2^{m-k}. \qquad \blacksquare$$

It is possible to represent the polynomial μ_1 using the coefficients of the polynomial μ_2. We start with the Taylor's formula for the polynomial $x \mapsto \mu_2(x_1, x)$:

$$\mu_2(x_1, x) = \sum_{k=0}^{m} \frac{\mu_2^{(k)}(x_1,0)}{k!}x^k. \tag{3}$$

Hence, identifying the corresponding coefficients in (3) and (1), we obtain

$$\frac{(-1)^k}{\mu_1(x_1)k!}\mu_1^{(k)}(x_1) = \frac{\mu_2^{(m-k)}(x_1,0)}{(m-k)!} \tag{4}$$

for $k = 0,\ldots,m$. Specially, for $k = m$, it follows that

$$\frac{1}{\mu_1(x_1)} = (-1)^m\mu_2(x_1,0). \tag{5}$$

Therefore

$$\mu_1(x) = \sum_{k=0}^{m} \frac{\mu_1^{(k)}(x_1)}{k!}(x - x_1)^k$$

$$= \frac{1}{\mu_2(x_1,0)} \sum_{k=0}^{m} \frac{(-1)^{m-k}}{(m-k)!} \mu_2^{(m-k)}(x_1,0)(x - x_1)^k. \tag{6}$$

2. RELATIONS BETWEEN THE COEFFICIENTS OF THE MINIMAL POLYNOMIALS

Suppose now that

$$P(x) = x^m + b_1 x^{m-1} + \cdots + b_m, \qquad b_m \neq 0,$$

$$Q(x) = x^m + c_1 x^{m-1} + \cdots + c_m, \qquad c_m \neq 0,$$

are two given polynomials. We shall answer the following

Problem. Do there exist numbers x_1, x_2 and an operator A such that the relations $P(x) = \mu_2(x_1, x)$, $Q(x) = \mu_2(x_2, x)$ hold?

It is sufficient to establish the existence of a polynomial

$$\mu_1(x) = x^m + a_1 x^{m-1} + \cdots + a_m$$

such that (1) holds, since then also the operator A exists for which μ_1 is the minimal polynomial.

Let us define $a_0 = b_0 = c_0 = 1$, and the vectors $a = [a_j]$, $\tilde{b} = [\tilde{b}_j]$ with

$$\tilde{b}_j = (-1)^j \frac{b_{m-j}}{b_m}, \qquad j = 0, \ldots, m, \tag{7}$$

and similarly for $\tilde{c}$.

If a polynomial P and a number x_1 are given, then the polynomial μ_1 must be given by (6).

PROPOSITION 1. *The coefficients of the polynomial μ_1 satisfy the equation*

$$T(x_1)a = \tilde{b}, \tag{8}$$

where the matrix T is defined by

$$[T(s)]_{i,j} = \begin{cases} \binom{m+i-j}{i-j} s^{i-j}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{9}$$

Proof. We use (1) and (5):

$$\begin{aligned} P(x) = \sum_{k=0}^{m} b_k x^{m-k} &= \frac{1}{\mu_1(x_1)} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \mu_1^{(k)}(x_1) x^{m-k} \\ &= (-1)^m b_m \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{j=k}^{m} \frac{\mu_1^{(j)}(0)}{(j-k)!} x_1^{j-k} x^{m-k} \\ &= \sum_{k=0}^{m} (-1)^m b_m \sum_{j=k}^{m} \binom{j}{k} x_1^{j-k} a_{m-j} x^{m-k}. \end{aligned}$$

Hence

$$\sum_{j=k}^{m} \binom{j}{k} x_1^{j-k} a_{m-j} = (-1)^{m-k} \frac{b_k}{b_m}, \qquad k = 0, \ldots, m,$$

which gives the equation (8), with T defined as in (9). ∎

PROPOSITION 2. *The mapping $s \mapsto T(s)$ defines a group of linear operators, with the multiplication as a group operation.*

Proof. It is obvious $T(0)=I$, and $[T(s)T(t)]_{i,j}=0$ for $i<j$. Suppose that $i \geqslant j$. Then

$$\begin{aligned}
[T(s)T(t)]_{i,j} &= \sum_{k=j}^{i} [T(s)]_{i,k}[T(t)]_{k,j} \\
&= \sum_{k=j}^{i} \binom{m+i-k}{i-k}\binom{m+k-j}{k-j} t^{i-k}s^{k-j} \\
&= \sum_{k=j}^{i} \binom{m+i-j}{i-j}\binom{i-j}{i-k} t^{i-k}s^{k-j} \\
&= \binom{m+i-j}{i-j}(t+s)^{i-j} = [T(s+t)]_{i,j}.
\end{aligned}$$

Thus, $T(s)^{-1}=T(-s)$. ∎

COROLLARY 2. *Let the polynomial $P(x)=\sum b_k x^{m-k}$, $b_0=1$, $b_m \neq 0$, and the number x_1 be given. Then there exists an operator A such that the relation $\mu_2(x_1, x)=P(x)$ holds. The coefficients of the minimal polynomial of the operator A are given by $a=T(-x_1)\tilde{b}$, where T and $\tilde{b}$ are defined by* (9) *and* (7).

THEOREM 2. *The answer to the Problem is affirmative iff the following relations hold*:

$$x_2 - x_1 = \frac{1}{m}\left(\frac{b_{m-1}}{b_m} - \frac{c_{m-1}}{c_m}\right), \tag{10}$$

$$\tilde{c} = T(x_2 - x_1)\tilde{b}. \tag{11}$$

Proof. By Corollary 2, we must have $a=T(-x_1)\tilde{b}$, $a=T(-x_2)\tilde{c}$, and these conditions are also sufficient. Hence, by Proposition 2, $\tilde{c} = T(-x_2)^{-1}T(-x_1)\tilde{b}=T(x_2-x_1)\tilde{b}$.

Let us prove (10). From (4) and (5) we obtain

$$\begin{aligned}
\frac{-(m-1)!c_{m-1}}{c_m} &= \mu_1^{(m-1)}(x_2) = \mu_1^{(m-1)}(x_1) + \mu_1^{(m)}(x_1)(x_2-x_1) \\
&= -\frac{(m-1)!b_{m-1}}{b_m} + m!(x_2-x_1),
\end{aligned}$$

and (10) follows. ∎

3. THE MINIMAL POLYNOMIAL OF THE ITERATED RESOLVENTS

We shall now derive a result similar to (2) for a sequence of the iterated resolvents, defined by

$$R_1(x_1) := (x_1 I - A)^{-1},$$
$$R_k(x_1,\dots,x_k) := (x_k I - R_{k-1})^{-1}, \qquad k = 2,3,\dots.$$

Denote by $x_n \mapsto \mu_n(x_1,\dots,x_n)$ the minimal polynomial of the resolvent R_{n-1}.

THEOREM 3. *The minimal polynomial* μ_n *is given by*

$$\mu_n(x_1,\dots,x_n) = \frac{1}{\mu_1(x_1)\mu_2(x_1,x_2)\dots\mu_{n-1}(x_1,\dots,x_{n-1})} \times \sum_{k=0}^{m} \frac{(-1)^k}{k!}\mu_1^{(k)}(0) B_n(x_1,\dots,x_n)^k B_{n-1}(x_2,\dots,x_n)^{m-k}, \tag{12}$$

where $\{B_n\}$ *is a sequence of polynomials defined by*

$$\begin{aligned} B_0 &= 1,\\ B_1(x_1) &= x_1, \\ B_n(x_1,\dots,x_n) &= x_n B_{n-1}(x_1,\dots,x_{n-1}) - B_{n-2}(x_1,\dots,x_{n-2}), \qquad n \geq 2. \end{aligned} \tag{13}$$

The first few polynomials B_n are

$$\begin{aligned} B_2(x_1,x_2) &= x_1x_2 - 1,\\ B_3(x_1,x_2,x_3) &= x_1x_2x_3 - x_1 - x_3,\\ B_4(x_1,x_2,x_3,x_4) &= x_1x_2x_3x_4 - x_1x_2 - x_1x_4 - x_3x_4 + 1,\\ B_5(x_1,\dots,x_5) &= x_1x_2x_3x_4x_5 - x_1x_2x_3 - x_1x_2x_5 - x_1x_4x_5 - x_3x_4x_5 \\ &\quad + x_1 + x_3 + x_5. \end{aligned}$$

Let us prove two lemmas.

LEMMA 1. $B_n(x_1,\dots,x_n)=x_nB_{n-1}(x_1,\dots,x_{n-2},x_{n-1}-1/x_n)$.

Proof. The statement of the lemma easily follows from the definition (13). ■

LEMMA 2. *For each* $0\leqslant k\leqslant m$ *we have*

$$B_n(x_1,\dots,x_n)^kB_{n-1}(x_2,\dots,x_n)^{m-k}$$
$$=\sum_{j=0}^{m}\frac{(-1)^j}{j!}x_n^{m-j}\frac{\partial^j}{\partial x_{n-1}^j}$$
$$\times\left\{B_{n-1}(x_1,\dots,x_{n-1})^kB_{n-2}(x_2,\dots,x_{n-1})^{m-k}\right\}. \tag{14}$$

Proof. Denote $P_n^k(x_1,\dots,x_n):=B_n(x_1,\dots,x_n)^kB_{n-1}(x_2,\dots,x_n)^{m-k}$. By Lemma 1 we have

$$P_n^k(x_1,\dots,x_n)=x_n^mB_{n-1}\left(x_1,\dots,x_{n-2},x_{n-1}-\frac{1}{x_n}\right)^k$$
$$\times B_{n-2}\left(x_2,\dots,x_{n-2},x_{n-1}-\frac{1}{x_n}\right)^{m-k}$$
$$=x_n^mP_{n-1}^k\left(x_1,\dots,x_{n-2},x_{n-1}-\frac{1}{x_n}\right).$$

By definition of the sequence $\{B_n\}$ we see that the function $x_n\mapsto B_n(x_1,\dots,x_n)$ is linear. Hence, $x_n\mapsto P_n^k(x_1,\dots,x_n)$ is a polynomial of degree m, for each k. Thus, if we expand the function $x_{n-1}\mapsto P_{n-1}^k(x_1,\dots,x_{n-2},x_{n-1}-1/x_n)$ in a Taylor series, we obtain

$$P_n^k(x_1,\dots,x_n)=x_n^m\sum_{j=0}^{m}\frac{1}{j!}\frac{\partial^j}{\partial x_{n-1}^j}P_{n-1}^k(x_1,\dots,x_{n-2},x_{n-1})\left(-\frac{1}{x_n}\right)^j$$
$$=\sum_{j=0}^{m}\frac{(-1)^j}{j!}x_n^{m-j}\frac{\partial^j}{\partial x_{n-1}^j}P_{n-1}^k(x_1,\dots,x_{n-1}),$$

and this is just the relation (14). ■

Proof of Theorem 3. We can deduce this theorem by induction. For $n = 2$ the statement is proved in Corollary 1. Suppose that the assertion is true for a number $n - 1$. By the same arguments as in the proof of the Theorem 1, we can write

$$\mu_n(x_1,\dots,x_n) = \frac{1}{\mu_{n-1}(x_1,\dots,x_{n-1})} \times \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial x_{n-1}^j} \mu_{n-1}(x_1,\dots,x_{n-1}) x_n^{m-j}.$$

By the induction hypothesis and Lemma 2, the right side is equal to

$$\frac{1}{\mu_{n-1}(x_1,\dots,x_{n-1})} \sum_{j=0}^{m} \frac{(-1)^j}{j!} x_n^{m-j} \frac{\partial^j}{\partial x_{n-1}^j} \frac{1}{\mu_1(x_1)\cdots\mu_{n-2}(x_1,\dots,x_{n-2})}$$

$$\times \sum_{k=0}^{m} \frac{(-1)^k}{k!} \mu_1^{(k)}(0) B_{n-1}(x_1,\dots,x_{n-1})^k B_{n-2}(x_2,\dots,x_{n-1})^{m-k}$$

$$= \frac{1}{\mu_1(x_1)\cdots\mu_{n-1}(x_1,\dots,x_{n-1})} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \mu_1^{(k)}(0)$$

$$\times \sum_{j=0}^{m} \frac{(-1)^j}{j!} x_n^{m-j} \frac{\partial^j}{\partial x_{n-1}^j}$$

$$\times \left\{ B_{n-1}(x_1,\dots,x_{n-1})^k B_{n-2}(x_2,\dots,x_{n-1})^{m-k} \right\}$$

$$= \frac{1}{\mu_1(x_1)\cdots\mu_{n-1}(x_1,\dots,x_{n-1})} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \mu_1^{(k)}(0)$$

$$\times B_n(x_1,\dots,x_n)^k B_{n-1}(x_2,\dots,x_n)^{m-k},$$

which proves the theorem. ■

REFERENCES

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